

Negative Conditional Entropy of Post-Selected States

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Abstract. Using Dirac complex distribution, and hence the statistics of weak measurements, we discuss a decomposition of “conditional state” of post-selected systems and introduce an entropic measure of information for them. In doing so we remark on the role of pre- and post- selection in the measurement of an ensemble. Conditional states are the quantum analogues of the conditional probabilities. We define them by selecting a particular condition in the measurement of a quantum system and studying a coarse grained set of events in the history of the state that ended in that particular condition. These states are different from what is known as conditional states in the literature [16, 6], in the sense that they are trace-1 operators and, by construction, they can be measured using weak measurements. We shall then define a conditional entropic measure based on these states, which as opposed to their classical counterparts, can have negative values. This is also the case even in the case of single state systems. This negative conditional entropy quantifies the amount of information in the post-selected ensembles, states which signify a non-separable class of histories of a quantum system.

1 Introduction

Dirac in his work introduced a complex phase-space distribution to make an “Analogy Between Classical and Quantum Mechanics”[8]. This distribution, however, is not limited to phase space. In fact any two operators with non-vanishing overlap between each of their eigenstates can be used to provide such a distribution. This is due to the fact that one can describe any quantum state of d -dimensional Hilbert Space with $d^2 - 1$ elements. Hence a space of any two such observables of a system would be sufficient to gain all information available in the state. Dirac distribution has been recently studied extensively from theoretical point of view [13, 12], and the experimental procedure for the direct measurement of that for a general quantum state has been given[17]. Hence we shall not review these concepts in depth in this paper. Our main results are Eqs. (9) and (12), where we utilise weak measurement statistics to give a decomposition of states which contain information about a single quantum system, relevant to a particular post-selection. Not only this decomposition signifies a particular pre- and post-selected ensemble, but also enables us to encompass a particular set of weak measurements to be done in the intermediate time. Then we use those states to define an entropy relevant to such information about histories. The remarkable fact about defining the conditional states of Eq. (9) is that they give a clear picture about the differences between conditional states in classical mechanics and their counterparts in quantum theory. Specifically, one can define an entropic measure, as we do in Eq. (12), that can have negative values, even in the case of single state systems.

2 Preliminaries

The Dirac distribution of a system, with a choice of operators A and B is given by

$$\Pr(a_m, b_n) = \text{Tr}[\rho A_m B_n] \quad (1)$$

where ρ is the density operator, a_m and b_n are the eigenvalues of the chosen operators A and B , and A_m and B_n are the projectors onto the corresponding eigenstates. Any choice of operators, A and B , will give a complete set of Dirac probabilities, as long as they have the same Hilbert space dimension as the state ρ to be described, and the eigenvectors of the two operators are mutually non-orthogonal and none of those eigenvectors are orthogonal to the original state ρ . We notice that any quantum state can be represented in a Dirac decomposition as follows

$$\rho = \sum_{m,n} \Pr(a_m, b_n) \frac{|a_m\rangle\langle b_n|}{\langle b_n|a_m\rangle}. \quad (2)$$

The Dirac distribution satisfies all the conditions of a classical Kolmogorov probability distribution, except that it is not a positive, real function; it is normalised and gives correct marginals,

$$\sum_n \Pr(a_m, b_n) = \sum_n \text{Tr}[\rho A_m B_n] = \text{Tr}[\rho A_m] \quad (3)$$

and

$$\sum_m \Pr(a_m, b_n) = \sum_m \text{Tr}[\rho A_m B_n] = \text{Tr}[\rho B_n], \quad (4)$$

and obeys the sum rule and the product rule. Dirac distribution is also compatible with Bayes law. It was shown that the negativity and complexness of this function is due to the non-commutativity of the quantum mechanical observables [13]. Hence, it should not be surprising that the probability distribution underlying a

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theory based on non-commuting observables is generally complex and outside the $[0, 1]$ interval. Furthermore, it has been recently shown by Morita *et al.* that to satisfy Gleason's theorem in explicitly time-symmetric models of quantum theory, without coarse graining, one has to adopt complex probability measures (For our comment on their results, see the appendix) [18]. This “fundamental probability distribution” has recently been rediscovered by Hartle in the context of histories-based quantum theories [10].

Hartle introduced the same distribution function to describe alternative histories. In the context of histories-based models of quantum theory, an individual history α of a quantum state is characterised by a sequence of projectors at each time

$$C_\alpha = P_n(t_n) \dots P_1(t_1). \quad (5)$$

Such chains are not generally projections themselves unless all of the members of the chain commute. In Hartle's work, the extended probability for a history is introduced as

$$\text{Pr}(\alpha) = \text{Re}[\langle \psi | C_\alpha | \psi \rangle]. \quad (6)$$

This is indeed the real part of the Dirac distribution in Eq. (1) for a pure initial state, where the two projectors used to describe the quantum state are replaced by the chain of projectors describing the history C_α ¹. In this context one may not be interested in the imaginary part. However, as was shown by Johansen, the imaginary part contains extra information that can be used to analyse the dynamics of the system [13].

One may notice the relevance of the aforementioned probabilities with weak values [1]. Given a projection operator, Π , into an arbitrary state, for an ensemble which is pre- and post-selected in $|\psi\rangle$ and $|\phi\rangle$ respectively, the weak values of the operator is given by

$$\Pi_w = \frac{\langle \phi | \Pi | \psi \rangle}{\langle \phi | \psi \rangle}. \quad (7)$$

This can be interpreted simply as the conditional Dirac distribution of the state $|\psi\rangle$, with the condition of being post-selected in the state $|\phi\rangle$. Defining $\rho_\psi := |\Psi\rangle\langle\Psi|$ and $\rho_\phi := |\Phi\rangle\langle\Phi|$, such a conditional probability is calculated using Bayes law as

$$\Pi_w = \text{Pr}(\Pi|\phi) = \frac{\text{Tr}[\rho_\psi \rho_\phi \Pi]}{\text{Tr}[\rho_\psi \rho_\phi]}. \quad (8)$$

Remembering that a history, in general, is described by a set of probabilities, Eq. (8) implies that pre- and post-selection in weak measurement is a method to separately measure different amplitudes, or histories, that

¹Here, for simplicity and without loss of generality we study the case for pure states. However, it was shown by Halliwell that Consistent Histories can be extended for mixed initial states [9]. The general mixed state measurement procedure using Dirac distributions was discussed by Lundeen and Bamber [17]. However, such extensions are not the concern here.

contribute to a quantum state, as is illustrated in Figure (1)². It has been discussed in the weak measurement literature that using such selections, one can ask counter-factual questions in quantum theory [2]. For instance, Eq. (8) asks what was the history of the system ρ_ψ , ending in ρ_ϕ , revealed by weak measurements, without destroying the state of the system before the post-selection measurement. We shall clarify this point further in the next section, when we calculate the states encoding information about these histories.

Nevertheless, to have an intuition about what these histories are, assume the 3-box problem [4]. Here a state is prepared, at time $t = 0$, in a superposition of three boxes A , B and C , *e.g.* $|\psi\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle)$. At a later time $t = 2$, the system is measured to be in some other state which is not orthogonal to $|\psi\rangle$, *e.g.* $|\phi\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle)$. Asking about the probability that the system was in box A at time $t = 1$ and ended in state $|\phi\rangle$ at time $t = 2$ is a particular question of history. Similarly, one can ask this question about boxes B and C . We have calculated an explicit example of the 3-box problem at the end of the next section. As discussed by Hartle, in the presence of interference these probabilities can take negative values. In this case, to put it in the language of Bayesian probability theory, the histories represent instances where one cannot settle a bet with a single basic measurement [10].

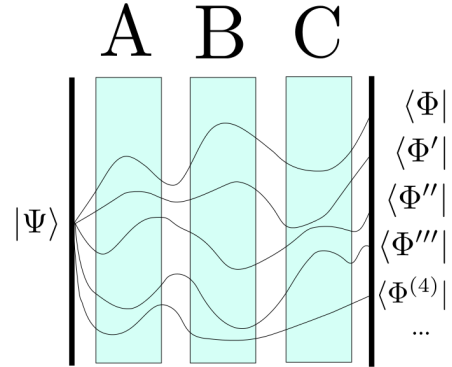


Figure 1: Different histories, distinguished by a unique post-selection in $|\phi^{(n)}\rangle$'s, and events A , B and C that can be counter-factually argued for, using weak measurements.

3 Quantum conditional states and conditional Entropy

The concept of “conditional state” is defined in different ways [15, 16, 6]. when used in an equation of classical conditional entropy, $H(A|B) = -\text{Tr}[\rho_{AB} \ln(\rho_{B|A})]$, it gives the correct quantum conditional entropy. However, the trace of such conditional states is generally not equal

²Note that this diagram is neither depicting what is normally referred to as “history” in the consistent histories literature, as the lines don't necessarily represent paths in spacetime, nor does it depict a history in the sense of the two-state formalism, as none of the axes represent time. Instead, this diagram intends to give a pictorial view of *histories of a state* by emphasizing that every post-selection singles out a different history to be summed to get the full picture of a quantum event.

to one, hence they cannot be regarded as physical states; one cannot directly measure them.

Conditional states as we shall define them in the following are measurable physical states, akin to conditional states of classical mechanics. As pointed out in Section 2, post-selection is a method to distinguish one alternative history from the others. Events in each particular history are characterised by the weak values – the results of weak measurements of the events with a pre- and post-selection for that particular history.

For instance in the figure (1), assuming a particle that can be in three different boxes, A , B and C . A particular choice of pre- and post-selection, take $|\psi\rangle$ and $|\phi\rangle$, determines a unique history and that history can be measured using weak measurements [4]. In other words this unique history with respect to a particular choice of pre- and post-selection means a specific choice of $|\psi\rangle$ and $|\phi\rangle$ gives a unique conditional probability in Eq. (8), with the condition “ending in state $|\phi\rangle$ ” being selected.

Using the concepts discussed above, we construct the density matrix corresponding to a conditional Dirac distribution,

$$\rho_{\psi|\phi} = \sum_{\text{history}} \text{Pr}(\text{history}|\phi) \frac{|\text{history}\rangle\langle\phi|}{\langle\phi|\text{history}\rangle}. \quad (9)$$

This is the density operator of all events with the condition that the state is post-selected in state $|\phi\rangle$. All possible final states $\{|\phi\rangle\}$ form a complete orthonormal basis set. Here $|\text{history}\rangle$ represents the quantum state of one possible history and the sum runs over all possible histories. For instance, if the system could be in the three boxes, A , B and C , then $|A\rangle$ is one possible history state $|\text{history}\rangle$, the others are $|B\rangle$ and $|C\rangle$. The state for this specific example is calculated later in this section in Eq. (18). The states thus defined are trace-one operators and can indeed be measured as they are, by construction, determined by weak measurements.

If we now multiply $\rho_{\psi|\phi}$ by probability $\text{Pr}(\phi)$ of the system ending in state $|\phi\rangle$ and sum Eq. (9) over all possible ϕ 's we retrieve the density operator of Eq. (2) which contains the full information about the system,

$$\rho = \sum_{\phi} \text{Pr}(\phi) \rho_{\psi|\phi}, \quad (10)$$

with $\text{Pr}(\phi) = \text{Tr}[\rho|\phi\rangle\langle\phi|]$ being the probability of the state being in the state $|\phi\rangle$ at the time of post-selection. This is what is expected from the analogy with classical probabilities.

One may note that the operator $\rho_{\psi|\phi}$ is not Hermitian. It has been argued in the weak measurement literature why this should not be a cause of concern [17]. We can now define a conditional entropy of post-selected systems in correspondence with classical conditional entropy. In classical information theory, the entropy of random variable X conditioned on selection of a particular instance y of random variable Y is $H(X|Y = y) = -\sum_x \text{Pr}(x|y) \ln \text{Pr}(x|y)$, where $\text{Pr}(x|y)$ is the conditional probability of x conditioned on the occurrence of y .

Then the classical conditional entropy $H(X|Y)$ is given by $H(X|Y) = \sum_y \text{Pr}(y) H(X|Y = y)$. Hence, similarly in quantum theory one can define an entropy of a state ψ conditioned on post-selection in one out of a possible set of $|\phi\rangle$'s as

$$S_c(\psi|\Phi = \phi) = -\frac{1}{2} \text{Tr}[\rho_{\psi|\phi} \ln(\rho_{\psi|\phi} \rho_{\psi|\phi}^\dagger)], \quad (11)$$

where $\rho_{\psi|\phi}^\dagger$ is the conjugate transpose of $\rho_{\psi|\phi}$. We note that this particular form of the entropy is the average $-\langle \ln((\rho_{\psi|\phi} \rho_{\psi|\phi}^\dagger)^{1/2}) \rangle$. The argument of the logarithm gives the Singular values of the operator $\rho_{\psi|\phi}$. The Singular Value Decomposition is used as a generalisation of diagonalisation to calculate the logarithm of the density operator, $\rho_{\psi|\phi}$. Such a generalisation is required due to the form of the density operator in Eq. (2), defined with the eigenstates of the A and B operators. Indeed for Hermitian operators, such as ρ_ψ the singular values and the eigenvalues are the same. At this point it worth noting that, similar to other entropies previously defined for quantum systems, one can define other objects which serve as a type of entropy. However, the choice made here is the most natural one. For instance one could define the function $S'_c(\psi|\Phi = \phi) = -\frac{1}{2} \text{Tr}[\rho_{\psi|\phi} \rho_{\psi|\phi}^\dagger \ln(\rho_{\psi|\phi} \rho_{\psi|\phi}^\dagger)]$ as such an information measure. However, this could not correspond to the average of the logarithm. The choice we made is indeed an average of the logarithm of the density operator of the system, decomposed in the appropriate basis. Finally, a quantum conditional entropy can be defined as

$$S_C(\psi|\Phi) = \sum_{\phi} \text{Pr}(\phi) S_c(\psi|\Phi = \phi), \quad (12)$$

This entropy, as can be verified in the following example later in this section, is a measure of information about a particular set of histories of a quantum mechanical state.

To find the upper and lower bound of the new quantum conditional entropy we need to take into account that the conditional states are obtained by summing over all histories. Hence they are independent of a particular history. To see this explicitly we use relation Eq. (8) to obtain

$$\text{Pr}(\text{history}|\phi) = \frac{\langle\phi|\text{history}\rangle\langle\text{history}|\psi\rangle}{\langle\phi|\psi\rangle} \quad (13)$$

and inserting this into Eq. (9) yields

$$\begin{aligned} \rho_{\psi|\phi} &= \sum_{\text{history}} \frac{|\text{history}\rangle\langle\text{history}|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle} \\ &= \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle} \end{aligned} \quad (14)$$

which is previously known equivalent form of the state to describe a pre- and post-selected ensemble. Inserting Eq. (14) into Eq. (11) we obtain

$$\begin{aligned} S_c(\psi|\Phi = \phi) &= -\frac{1}{2} \frac{\langle\phi|\ln(\rho_{\psi|\phi} \rho_{\psi|\phi}^\dagger)|\psi\rangle}{\langle\phi|\psi\rangle} \\ &= \ln|\langle\phi|\psi\rangle| \end{aligned} \quad (15)$$

Similar to the fact that several different ensembles can be described using the same density operator, several different histories can be described using the same conditional density operator. Remarkably, the path independence of conditional states holds for longer sequences of questions about the history of a state, such as the general one in Eq. (5). This is why, in general, one can calculate quantum mechanical probabilities using the path integral formalism, without the path being the real histories of a particle, moving in spacetime.

Thus, even though we ask questions about the history using weak measurement and the conditional Dirac distribution we obtain unique probabilities. The conditional state quantum is only sensitive to the specific choice of post-selection. Hence, we can exploit this form of the state, and apply it in Eq. (12), to calculate the entropy bounds as follows

$$S_C(\psi|\Phi) = \sum_{\phi} |\langle\phi|\psi\rangle|^2 \ln|\langle\phi|\psi\rangle| \quad (16)$$

Now we can see that S_C has a negative lower bound of $\frac{1}{d} \ln \frac{1}{\sqrt{d}}$, where d is the Hilbert Space dimension of the system. This lower bound is reached when all $|\phi\rangle$'s have the same overlap with $|\psi\rangle$, *i.e.* in the case of equidistribution of post-selected states. S_C is bounded from above by the von Neumann entropy of the system ρ itself (Eq. (2)).

Example: 3-box problem [4]

Assume there are three boxes, A, B and C, where a quantum particle can be found. If a system is prepared in the state $|\psi\rangle = 1/\sqrt{3}(|A\rangle + |B\rangle + |C\rangle)$, if later the state is selected to be in $|\phi\rangle = 1/\sqrt{3}(|A\rangle + |B\rangle - |C\rangle)$, one can calculate the conditional probabilities using Eq. (8) as

$$\begin{aligned} \Pr(A|\phi) &= 1, \\ \Pr(B|\phi) &= 1, \\ \Pr(C|\phi) &= -1, \end{aligned} \quad (17)$$

which is the same the results of the weak values of 3-box problem, preselected in the state $|\psi\rangle$ and post-selected in the state $|\phi\rangle$. The conditional state of histories, with condition $|\phi\rangle$ being selected, defined in the Eq. (9) is

$$\rho_{\psi|\phi} = \sum_{box \in \{A, B, C\}} \Pr(box|\phi) \frac{|box\rangle\langle\phi|}{\langle\phi|box\rangle}. \quad (18)$$

Once such a state is measured and/or calculated, one can work out the conditional entropy of the system with selected state, $|\phi\rangle$, given in Eq. (11), to be $S_C(\psi|\Phi = \phi) = -\ln 3$. Now to calculate the conditional entropy of Eq. (12), given that the Hilbert space dimension of the system is three, one needs to perform the same calculation as above for two other post-selected states, $|\phi'\rangle$ and $|\phi''\rangle$ to have sufficient number of Dirac probabilities to completely describe the state ρ_{ψ} . The only restrictions of the choice of these two post-selected states are that they need to be mutually non-orthogonal to the state of each box, $|A\rangle, |B\rangle$ and $|C\rangle$, and they need, together with our

original state $|\phi\rangle$, to span the Hilbert space of the state $|\psi\rangle$. Take these states to be

$$|\phi'\rangle = \frac{1}{\sqrt{3}}|A\rangle + \frac{-3-\sqrt{3}}{6}|B\rangle + \frac{-3+\sqrt{3}}{6}|C\rangle \quad (19)$$

and

$$|\phi''\rangle = \frac{1}{\sqrt{3}}|A\rangle + \frac{3-\sqrt{3}}{6}|B\rangle + \frac{3+\sqrt{3}}{6}|C\rangle. \quad (20)$$

The same calculation of Eq. (11) as in the case of the state, with post-selection $|\phi\rangle$, repeated for $|\phi'\rangle$ and $|\phi''\rangle$ gives $S_c(\psi|\Phi = \phi') = -\log_3 4.10$ and $S_c(\psi|\Phi = \phi) = -\log_3 1.10$. With probabilities $\Pr(\phi) = 0.11$, $\Pr(\phi') = 0.06$ and $\Pr(\phi'') = 0.83$, the conditional entropy Eq. (12) becomes $S_C(\psi|\Phi) = -0.26$, with the logarithm being calculated in base 3.

As we see above, conditional entropies can have negative values in the presence of quantum interference. This is what we expect. To clarify this concept, once again we emphasise that the post-selection gives a class of histories, or amplitudes, where the particle ended in the state $|\phi\rangle$. This is partial information about a “non-separable” part of a system. Hence producing negative partial entropy in the quantum system, akin to the negative conditional entropy in higher dimensions. Nevertheless, this is just measuring a particular part of the amplitude of a single system as opposed to one separate particle in a non-separable bipartite system.

Conclusion In this work we have calculated conditional states as operators in two equivalent forms. One in a Dirac distribution decomposition Eq. (9) and the more condensed form of Eq. (14). The choice of Dirac decomposition serves as a unified formalism to infer statements about particular retrodictive reasoning of quantum events, whereas the latter form is used in the case of calculations which only concerns about the results of the pre- and post-selection, regardless of particular histories. We interpreted them as states summarising information about a distinct set of histories of a post-selected quantum states. These states, as opposed to the conditional states in [6, 16] are trace-1 operators and are the quantum analogue of classical conditional probabilities in the sense that one particular condition is selected and the probability of other relevant events are studied. However, we also showed that these states have properties beyond their classical counterparts due to non-commutativity in quantum mechanics. Most notably, conditional quantum entropies as defined in this work can have negative values, similar to non-separable states in multi-partite systems. Previously, negative entropies were not defined for the case of single particles. We shall discuss the single-particle non-locality of these states in a forthcoming correspondence. As a final point of the discussion we wish to mention that, similar to other entropies previously defined for quantum systems, one can define other entropic measures for post-selected ensembles. However, as we discussed in the paper, ours is the most natural choice among others.

Acknowledgements We thank Jeff Lundeen and Holger Hofmann for their helpful comments and stimulating discussions.

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A Appendix: On the Interpretation of Complex Probabilities

In this work we have shown that complex probabilities, emerged as a consequence of non-commutativity of quantum mechanics, can be used as a calculational tool to analyse the results of histories and their measurements. However, we left the question of intelligibility of these extended probabilities open, as this is not intended to be

the focus of this research. Nonetheless, a point regarding interpretation of them can be helpful in discovering the fundamental role they play as an underlying logical structure of the nature.

Several interpretations of probabilities has been suggested so far [11]. The Frequentist and the Bayesian interpretation are among the most received ones, with the Frequentist interpretation taking the lead by far in terms of its popularity. The reason for such a wide use is its success in offering a logical structure that can be used to reason about the classical world as we know it. Nevertheless, that does not mean this interpretation is free of pitfalls, even in the realm of classical mechanics. Laplace, a champion of these classical probabilities described [14] probabilities as

“The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible.”

This interpretation, being the classical one from which the Frequentist interpretation descends, has apparent fundamental issues. To determine such probabilities, one needs to make measurement of infinite sequence of events. If the probabilities are defined to describe the physical world, then one cannot try a random event infinite number of times. Hence that probability can never be determines. This cannot be modified by simply limiting the number of trials to just a large number. This is due to the fact that this method gives different probabilities for different sequence of trials. Alternatively, de Finetti suggested that probabilities are strategies for making fair bets [7]. It has been argued why, in classical mechanics, de Finetti’s interpretation of probabilities are real positive values [10]. The reason is then the bookie, to adopt the language of subjective probability, can offer a ‘Dutch Book’ by some choice of stakes. However, this problem does not occur in the case of quantum complex probabilities, since these probabilities emerge in the presence of interference. Hence the non-commutativity of operators does not allow the bookie to offer a Dutch Book, as the bet cannot be settled by a single basic measurement on one particle. Thus, there is no guarantee on the result of the bet, each time the measurement is performed. Even in the case of weak measurement, the measurement result from a single particle has an uncertainty greater than what is required to settle a bet. Therefore these probabilities are compatible with the Bayesian interpretation. Moreover, relaxing mathematical axioms, to come up with self-consistent mathematical frameworks

that best describe the physical phenomena, is not unfamiliar to the contemporary science. The same way it is understood that the underlying geometry of nature in presence of gravity is non-Euclidean, one should not be reluctant to adopt non-Kolmogorovean probability theory as an underlying mathematical structure of nature at the quantum level.

On the other hand, some care must be taken on the extent to which we interpret these probabilities. As we have shown in this paper, these functions can be used to describe the dynamics of a quantum systems in alternative histories. The histories that end in a particular state, distinguished by post-selection. However, states are not often naturally post-selected in a state, *i.e.* quantum states do not have one single deterministic history. They are composed of several alternative histories, each ending in a different state undetermined in advance, over which we sum, proportional to the probability of each post-selection, to describe the complete quantum states. Hence, in interpreting complex probabilities, one has to bear in mind that this is a consequence of having joint distribution of two interfering states, or alternative histories in one single state. Thus, in the two-state formulations of quantum mechanics [5], and two-state probability measures [18], while it is correct that quantum mechanics, on its own, is time-symmetric and the evolution backwards in time is described by the dual states to the ones evolving forwards in time. Nonetheless, regardless of the future of a state, one can always post-select according to a different alternative history, and consequently get a different probabilities and different measurement results, because they describe different physical situations. For one particular instance the “Fate of the Universe” [3], need not be post-selected in a particular state and our cosmological observations need not be a reflection of such post-selection. Hence, we conclude that the two-state formalism and pre- and post- selection are methods to put boundary conditions on a set of histories to single them out, as opposed to giving description of the whole system and its histories by only one choice of post-selection.